MTH314: Discrete Mathematics for Engineers

Lecture 5: Mathematical Principles

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Each of 26 people is given a set of 9 balls numbered from 1 to 9 as pictured. Each of them can choose at least one and most three of them. Show that there must be two people with the same sum of numbers on the balls they chose.

Review: Pigeonhole Principle

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Solution: The possible sums are integers. The smallest possible sum is either 0 or 1, depending on if we're allowing the people to not choose any balls. The largest is $9+8+7=24$. Therefore, there are either 24 or 25 possible sums and 26 people, so there must be some two people with the same sum. □

Review: Induction

For the Fibonacci Sequence $\{a_n \mid n \in \mathbb{N}\}$, defined by:
$a_0 = 0$
$a_1 = 1$
$a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$,
show that:

$$\forall n \in N,\ a_0 + a_1 + \cdots + a_n = a_{n+2} - 1$$

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Base case: for $n = 0$

Inductive step: assume that $a_0 + a_1 + \cdots + a_n = a_{n+2} - 1$
("inductive hypothesis")

Show that $a_0 + a_1 + \cdots + a_n + a_{n+1} = a_{n+3} - 1$

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Base case: for $n = 0$,
LHS=$a_0 = 0$ and
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By the principle of mathematical induction,
$\forall n \in \mathbb{N},\ a_0 + a_1 + \cdots + a_n = a_{n+2} - 1.$

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By the principle of mathematical induction,
$\forall n \in \mathbb{N},\ a_0 + a_1 + \cdots + a_n = a_{n+2} - 1.$

This is an example of how we prove a property of a recurrence relation.

Recurrence Relations

(From now on we may implicitly assume n or k is a natural number.)

$$\forall k \geq 0,\ s_{k+1} = 3s_k - 1$$

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Do these things mean the same thing?

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Einstein's field equation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} - \Lambda g_{\mu\nu}$$

If the cosmological constant term is on the left, we think of it as dark matter. If it's on the right, we think of it as dark energy.

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$$\forall k \geq 1,\ s_{k+1} = 3s_k - 1$$

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Clearly NOT the same thing as the ones above.

Closed formula

A closed formula of a recurrence relation is an expression for the nth term in terms of the index n, not the previous terms.

Example:

- $a_0 = 0$ Recursive form
  $a_n = a_{n-1} + 1, \ \forall n \geq 1$

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- $a_0 = r$ — Recursive form
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$$r, \ r+s, \ r+2s, \ r+3s, \ldots$$

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Guess: $a_n = r + ns$. Let's prove it.

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  Proof: base case: $a_0 = r = r + 0r$ ✓
  Suppose that $a_n = r + ns$. Then:
  $a_{n+1} = a_n + s = r + ns + s = r + (n+1)s$ ✓

  $a_n = r + ns$ Closed formula

Closed formula

- $a_1 = 1$
  $a_n = 2a_{n-1} + 1, \ \forall n \geq 2$

Recursive form

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$$1, \ 3, \ 7, \ 15, \ 31, \ 63, \ldots$$

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Suppose that $a_n = 2^n - 1$. Want: $a_{n+1} = 2^{n+1} - 1$.

$a_{n+1} = 2a_n + 1$

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$a_n = 2^n - 1$ Closed formula

LHRR

An equation of the form

$$a_k = Aa_{k-1} + Ba_{k-2},$$

where A, B are any reals (not depending on k), is called a *linear, homogeneous, recurrence relation with constant coefficients, of degree 2.* We'll call it a "degree-2 LHRR."

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Example:

$$a_k = -5a_{k-1} + \frac{1}{2}a_{k-2}$$

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- $5a_{n-1} + na_{n-2}$
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- $5a_{n-1} - 2a_{n-1} + 7$

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The following are NOT LHRR:

- $5a_{n-1} + na_{n-2}$ NOT CONSTANT COEFFICIENTS
- $5\sqrt{a_{n-1}} + a_{n-2}$ NOT LINEAR
- $5a_{n-1} - 2a_{n-1} + 7$ NOT HOMOGENEOUS

Characteristic equation (of an LHRR)

A recurrence relation of the form

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has a *characteristic equation*

$$t^2 = At + B.$$

We can find the closed formula of these sequences by finding the roots of this equation.

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We can find the closed formula of these sequences by finding the roots of this equation. Suppose that $a_0 = \alpha_0$, $a_1 = \alpha_1$ where $\alpha_{0,1}$ are some real numbers. Then we want to find constants $C,\ D$ such that:

$$\alpha_0 = C + D$$

$$\alpha_1 = Cs + Dr$$

Where $r,\ s$ are the roots of $t^2 = At + B$.

Characteristic equation (of an LHRR)

If the roots of $t^2 = At + B$ are two distinct real numbers r and s, then the closed form of a_k is:

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. Example: the Fibonacci Sequence.

- Write down the characteristic equation of the Fibonacci Sequence.
- Find the roots $r, \ s$ of this equation.
- Find constants $C, \ D$ such that:

$$a_0 = C + D$$

$$a_1 = Cs + Dr$$

Where $r, \ s$ are the roots of $t^2 = At + B$.

- The closed formula is:

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- Find constants $C,\ D$:

$$0 = C + D$$

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- Find constants $C, \ D$:
$$0 = C + D$$
$$1 = C\frac{1+\sqrt{5}}{2} + D\frac{1-\sqrt{5}}{2}$$
From the first equation we get $D = -C$. Substitute that into the second equation:
$$1 = C\left(\frac{2\sqrt{5}}{2}\right) = C\sqrt{5}, \text{ so } C = \frac{1}{\sqrt{5}}$$

Closed formula of the Fibonacci Sequence

We know that $r = \frac{1+\sqrt{5}}{2}$ and $s = \frac{1-\sqrt{5}}{2}$,
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a_0 ✓

a_1 ✓

$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^2 + \frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^2$

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a_0 ✓
a_1 ✓
$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^2 + \frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1}{\sqrt{5}}\frac{1+2\sqrt{5}+5}{4} + \frac{-1}{\sqrt{5}}\frac{1-2\sqrt{5}+5}{4}$

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$$= \frac{1}{4\sqrt{5}}(4\sqrt{5}) = 1 = a_2 \checkmark$$

Closed formula of the Fibonacci Sequence

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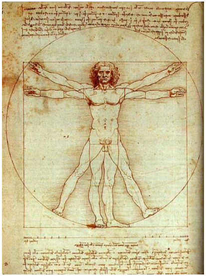
$$=\frac{1}{8\sqrt{5}}(16\sqrt{5})=2=a_3 \checkmark$$

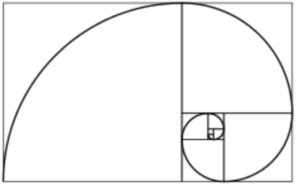
Golden Ratio

$$\varphi = \frac{1+\sqrt{5}}{2}$$

is called the *golden ratio*. It's the number such that:

$$\text{If } \frac{A+B}{A} = \frac{B}{A}, \text{ then } \frac{B}{A} = \varphi$$





Distinct Roots vs Single Root

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Distinct Roots vs Single Root

We've assumed that $t^2 = At + B$ has 2 distinct real roots $r,\ s$. What if it has a double root r, like $t^2 = 4t - 4$? (In which case $r = 2$.)

Then the closed form of a_k is:

$$a_k = (Ck + D)r^k$$

So we need to find $C,\ D$ that satisfy

$$D = a_0$$

$$(C + D)r = a_1$$

Since $a_0 = (C \times 0 + D)r^0 = D$,
$a_1 = (C \times 1 + D)r^1 = (C + D)r$.

Distinct Roots vs Single Root: Example

If $t^2 = At + B$ has a double root r, the closed form of a_k is:

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So we need to find $C, \ D$ that satisfy

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Distinct Roots vs Single Root

Theorem (Distinct Roots Theorem)

Let $a_0 = \alpha_0$, $a_1 = \alpha_1$ and $a_k = Aa_{k-1} + Ba_{k-2}$ for all $k \geq 2$ define a recursive sequence. Then if $t^2 = At + B$ has two distinct real roots r, s, then

$$a_k = Cr^k + Ds^k, \ \forall k \in \mathbb{N}$$

for some real constants C, D.

We find C, D by solving this equation for a_0 and a_1.

Distinct Roots vs Single Root

Theorem (Single Root Theorem)

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Structural Induction (Recursion)

For some sets, we can define the membership in the set recursively.

Example: the set of natural numbers $\mathbb{N}$ can be defined as follows:

$$0 \in \mathbb{N}$$

$$n \in \mathbb{N} \rightarrow n + 1 \in \mathbb{N}$$

$\mathbb{N}$ contains nothing else.

Then every element is in the set if and only if it is added to the set at some point by this procedure.

If you run this program, it will never finish no matter how long you leave it running for. But for every particular number, that number will be added in a finite time!

Structural Induction/Recursively Defined Sets

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↑ THINK ABOUT THIS FOR A BIT

Example:

Define a set A of binary strings in a following way.

- The empty string $\emptyset$ and $\mathbf{0}$ are both in A.
- If a binary string s is in A, then the string $s1$ obtained from s by attaching 1 at the end is in A, and the string $s10$ obtained from s by attaching 10 at the end.
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Strings in A:
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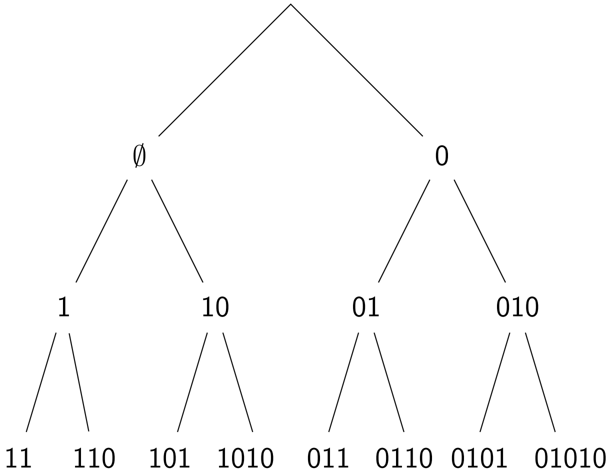
Strings in A:
$\emptyset$, 0, 01, 010, 011, 0110, 0101, 01010, ...

We can think about the set as having the strings added to it in steps. First, at step 1 $\emptyset$, 0. Then at each step, what we create from those created in the previous step by attaching either 1 or 10 to each. So at kth step we add 2^k strings.

Structural Induction/Recursively Defined Sets



Structural Induction/Recursively Defined Sets

Want to prove that the set A of strings built up from $\{\emptyset, 0\}$ by adding "1" and "10" pieces is the same as the set B of strings that have no "00."

Proof outline. Part 1: $A \subseteq B$. We want to show that no string in A has a "00."

Part 2: $B \subseteq A$. We want to show that any string that has no "00" can be built up from $\{\emptyset, 0\}$ by adding "1" and "10" pieces.