A few words on Assignment 3

Solutions to question 2:

a) $\{4, 5, 6, 7, 8, 9, 10\}$

b) $\{-2, -1, 0, 1, 2, 3, 4, 5\}$

c) $\{-2, -1, 0, 1, 2, 3, 4, 5\}$

d) $(0, 4] \cup (5, 6]$

e) $(-1, 0]$

A few words on Assignment 3 - question 3a

Prove or disprove that $A - B^C = A \cap B$.

Proof: Part 1: $A - B^C \subseteq A \cap B$.
Suppose that $x \in A - B^C$. Then $x \in A$, and $x \notin B^C$. Then since $x \notin B^C$, we know that $x \in B$. So together with $x \in A$ this implies $x \in A \cap B$. So since every element x such that $x \in A - B^C$ is also an element of $A \cap B$, we conclude that $A - B^C \subseteq A \cap B$.

Part 2: $A \cap B \subseteq A - B^C$.
Suppose that $x \in A \cap B$. Then $x \in A$ and $x \in B$, which also means $x \notin B^C$. Then in particular $x \in A$ and $x \notin B^C$, so $x \in A - B^C$. We conclude that since any element of $A \cap B$ is an element of $A - B^C$, $A \cap B \subseteq A - B^C$.

Part 1 and Part 2 together imply $A - B^C = A \cap B$. $\square$

A few words on Assignment 3 - question 4a)

Recall that the *power set* of a set A, $\mathcal{P}(A)$, is the set of all subsets of A. So an element of $\mathcal{P}(A)$ is a set S such that $S \subseteq A$. We want to show is the following is true:

$$\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$$

Proof: Part 1, we want to show that $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

So suppose that $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then $S \in \mathcal{P}(A)$, so $S \subseteq A$, and also $S \in \mathcal{P}(B)$, so $S \subseteq B$. So any element $x \in S$ must be an element of A and also an element must be of B.

So in particular, $x \in A \cap B$, and so $S \subseteq A \cap B$ and $x \in \mathcal{P}(A \cap B)$. We conclude that $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

A few words on Assignment 3 - question 4a)

Part 2, we want to show that $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.

Suppose that $S \in \mathcal{P}(A \cap B)$, so $S \subseteq A \cap B$. Then in particular $S \subseteq A$ and $S \subseteq B$, so $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$. We conclude that $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$, and the two parts together imply $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$. $\square$

A few words on Assignment 3 - question 4d)

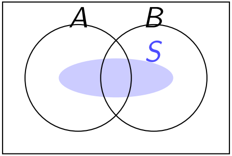
Consider the claim that $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

We can show that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq P(A \cup B)$. Is $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$ then $S \subseteq A$ or $S \subseteq B$, or both. If $S \subseteq A$, then every $x \in S$ belongs to A, so every $x \in S$ belongs to $A \cup B$ and so $S \subseteq A \cup B$.

Otherwise, if $S \not\subseteq A$ then necessarily $S \subset B$. Then, again, every element $x \in S$ is also in B, and so $x \in A \cup B$. Either way, $S \subseteq A \cup B$ and so we conclude that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

A few words on Assignment 3 - question 4d)

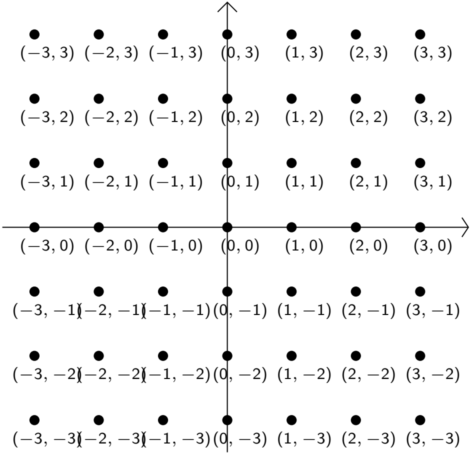
Here is a picture of a set S that is a subset of $A \cup B$, but does not belong to $\mathcal{P}(A) \cup \mathcal{P}(B)$:



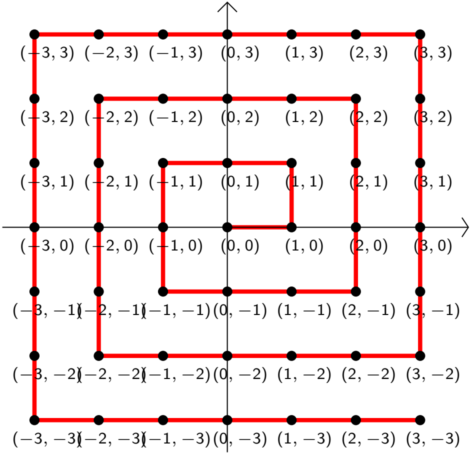
If we're trying to disprove something, we only need a counterexample. So take $A = \{0, 1\}$ and $B = \{3\}$. Then $S = \{1, 3\} \in \mathcal{P}(A \cup B)$, but $S \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

We conclude that $\mathcal{P}(A) \cup \mathcal{P}(B) \not\subseteq \mathcal{P}(A \cup B)$.

A few words on Assignment 3



A few words on Assignment 3



MTH314: Discrete Mathematics for Engineers

Lecture 4: Induction

Dr Ewa Infeld

Ryerson Univesity

25 January 2017

The Principle of Mathematical Induction

$$\forall n \in \mathbb{N},\ P(n) \rightarrow P(n+1)$$

$$P(0)$$

$$\therefore \forall n \in \mathbb{N},\ P(n)$$

If we know that "if it's true for a natural number n then it must be true for the next natural number" and we know that "it's true for 0" then we also no that "it's true for every natural number.

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Example: Let's prove that for all $n \geq 1$,

$$2^0 + 2 + 1 + \cdots + 2^{n-1} = 2^n - 1.$$

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Inductive step: suppose true for n. Show for $n + 1$. Which means that our assumption is:

$$2^0 + 2^1 + \cdots + 2^{n-1} = 2^n - 1 = 2^n - 1$$

and we want to arrive at:

$$2^0 + 2^1 + \cdots + 2^{n-1} + 2^n = 2^{n+1} - 1.$$

The Principle of Mathematical Induction

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$$\begin{aligned} &= 2^n + 2^n - 1 \\ &= 2^n + 2^0 + 2^1 + \cdots + 2^{n-1} \\ &= 2^0 + 2^1 + \cdots + 2^{n-1} + 2^n \end{aligned}$$

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Therefore, by the principle of mathematical induction, $2^0 + 2 + 1 + \cdots + 2^n = 2^{n+1} - 1$. □

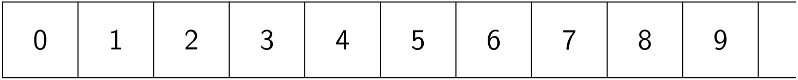
Visual

A visual way to think about it.

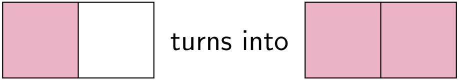
0	1	2	3	4	5	6	7	8	9	

Visual

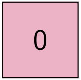
A visual way to think about it.



Inductive step:



Base case:



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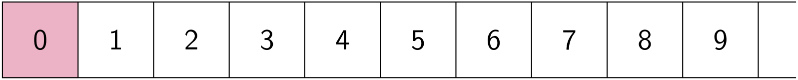

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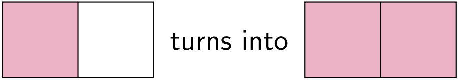

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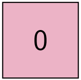

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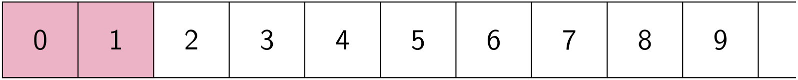

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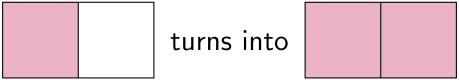

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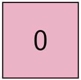

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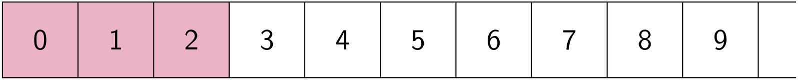
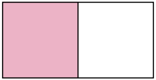
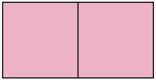


Inductive step:



turns into



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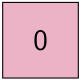

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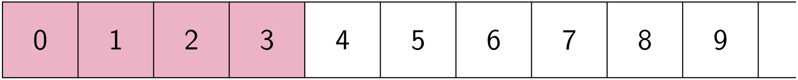

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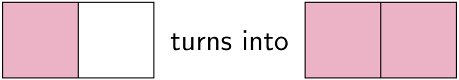

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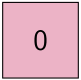

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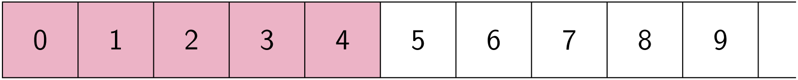

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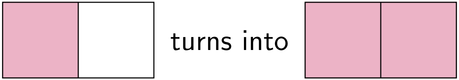

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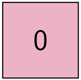

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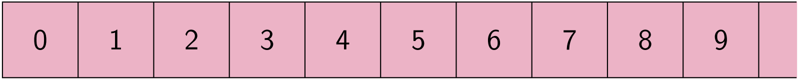

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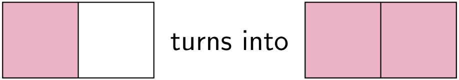

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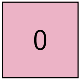

Exercise 1

Prove that for all natural numbers $n \geq 1$:

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

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Base case: check for $n = 1$.

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Inductive step: assume $1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$, show:

$$1 + 2 + 3 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2}$$

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Inductive step: assume $1+2+3+\cdots+(n-1)+n=\frac{n(n+1)}{2}$,
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LHS$=1+2+3+\cdots+n+(n+1)$

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LHS$=1+2+3+\cdots+n+(n+1)$

$$=\frac{n(n+1)}{2}+(n+1)$$

$$=\frac{n(n+1)+2(n+1)}{2}$$

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$$\text{LHS} = 1+2+3+\cdots+n+(n+1)$$

$$= \frac{n(n+1)}{2}+(n+1)$$

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LHS$=1+2+3+\cdots+n+(n+1)$

$=\frac{n(n+1)}{2}+(n+1)$

$=\frac{n(n+1)+2(n+1)}{2}$

$=\frac{(n+2)(n+1)}{2}=RHS$

By the principle of mathematical induction we have shown that for all natural numbers $n\geq 1$:

$$\sum_{k=1}^{n} k = 1+2+\cdots+n=\frac{n(n+1)}{2}$$

Exercise 2

Use induction to prove that for all natural numbers $r \neq 1$, and natural numbers $n \geq$:

$$\sum_{k=0}^{n} r^k = r^0 + r^1 + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

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Base case: r can be any real number, $n = 1$.

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$$\sum_{k=0}^{n} r^k = r^0 + r^1 + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

Base case: r can be any real number other than 1, $n = 1$.

$$r^0 + r^1 = 1 + r = \frac{(1 + r)(r - 1)}{r - 1}$$

Exercise 2

Use induction to prove that for all natural numbers $r \neq 1$, and natural numbers $n \geq$:

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$$\frac{(r + 1)(r - 1)}{r - 1} = \frac{r^2 - 1}{r - 1}$$

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$$r^0 + r^1 = 1 + r = \frac{(1 + r)(r - 1)}{r - 1}$$

$$\frac{(r + 1)(r - 1)}{r - 1} = \frac{r^2 - 1}{r - 1} \quad \checkmark$$

Exercise 2

Inductive step. Assume:

$$\sum_{k=0}^{n} r^k = r^0 + r^1 + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

And show that:

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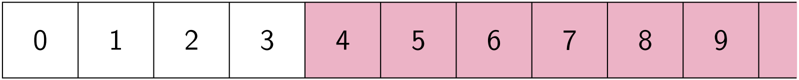
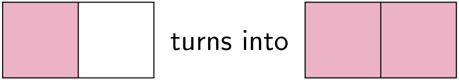
$$= \frac{r^{n+2}-1}{r-1} = RHS$$

By the principle of mathematical induction... □

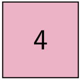
Modification: starting further down the line.



Inductive step:



Base case:



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Inductive step: assume that $2^n < n!$. Want: $2^{n+1} < (n+1)!$.

LHS$=2^n \times 2 < n! \times 2 < n! \times (n+1) =$RHS,
because $n + 1 > 4$ and so $n + 1 > 2$.

By the principle of mathematical induction, for all natural numbers $n \geq 4$, $2^n < n!$ $\square$

Another modification: "strong" induction.

Strong induction: we now need a statement to be true for all natural numbers up to n, to argue it's true for $n+1$.

0	1	2	3	implies	4

Need them all!

$$\forall n \in \mathbb{N},\ (\forall k \in \mathbb{N} \text{ such that } \ k \leq n,\ P(n)) \rightarrow P(n+1)$$

$$P(0)$$

$$\therefore \forall n \in \mathbb{N},\ P(n)$$

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Need them all! (or sometimes a subset)

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$$P(0)$$

$$\therefore \forall n \in \mathbb{N},\ P(n)$$

$P(n)$ is still true for all n, and base case doesn't change. But the inductive step uses a different argument!

Strong induction: Example

Let $n \geq 12$ be a natural number.

$P(n)$: an n-cent postage can be made up of 3-cent and 7-cent stamps.

Proof outline: We will check $P(12)$, $P(13)$, and $P(14)$. Then we will use $P(n-2)$, $P(n-1)$, $P(n)$ to show $P(n+1)$.

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Base case:
$12 = 3 + 3 + 3 + 3$
$13 = 3 + 3 + 7$
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Base case:

$12 = 3 + 3 + 3 + 3$

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Inductive step:

$P(n+1) = P(n-2) + 3$

$P(n+2) = P(n-1) + 3$

$P(n+3) = P(n) + 3$

Fibonacci Numbers

The Fibonacci sequence is a sequence of numbers $a_0, \ a_1, \ a_2, \ \ldots$ that starts with $a_0 = 1, \ a_1 = 1$ and then every next number is the sum of the previous two.

$$1, \ 1, \ 2, \ 3, \ 5, \ 8, \ 13, \ 21, \ 34, \ 55, \ 89, \ 144, \ \ldots$$

$$\begin{aligned} a_2 &= 1 + 1 = 2 \\ a_3 &= 1 + 2 = 3 \\ a_4 &= 2 + 3 = 5 \\ &\vdots \end{aligned}$$

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$a_n = a_{n-1} + a_{n-2}$ for all natural numbers $n \geq 2$.

It's an example of recurrence relation. The following information is what you need to define the whole sequence:
$a_0 = 1$, $a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$ for all natural numbers $n \geq 2$.

Recurrence Relations

A recurrence realtion is an equation that recursively defines a sequence of numbers, which means that every next term in the sequence are defined in relation to some previous terms.

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Example: Fibonacci Numbers
$a_0 = 1$
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Example: Exercise 3 on the worksheet

Exercise 3

Define a sequence as follows: $a_1 = 1$
$a_2 = 1$
$a_3 = 1$
$a_n = a_{n-1} + a_{n-2} + a_{n-3} \; \forall n \geq 4$

$$1,\ 1,\ 1,\ 3,\ 5,\ 9,\ 17,\ 31,\ \ldots$$

Want to use strong induction to show that $\forall n \geq 1$:

$$a_n \leq 2^n$$

Exercise 3

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$$1, \ 1, \ 1, \ 3, \ 5, \ 9, \ 17, \ 31, \ \ldots$$

Want to use strong induction to show that $\forall n \geq 1$:

$$a_n \leq 2^n$$

Base case: looks like we'll need 3 terms.
$a_1 = a_2 = a_3 = 1 \leq 2^1 \leq 2^2 \leq 2^3$

Exercise 3

Inductive step: suppose that $a_n \leq 2^n$
$a_{n+1} \leq 2^{n+1}$
$a_{n+2} \leq 2^{n+2}$
We want to show that:

$$a_{n+3} \leq 2^{n+3}$$

LHS$= a_n + a_{n+1} + a_{n+2} \leq 2^n + 2^{n+1} + 2^{n+2}$

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So we conclude that $a_{n+3} \leq 2^{n+3}$.'

By the principle of strong mathematical induction, $a_n \leq 2^n$ for all $n \geq 1$. □