**Lecture 8: Worksheet** **Math 20 Fall 2014, Dartmouth College**

## 1 Continuous Conditional Probability

If $X$ is a continuous random variable with density function $f(x)$, and $E$ is an event, the conditional density function of $f(x|E)$ is given by:

$$f(x|E) = \begin{cases} f(x)/P(E) & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

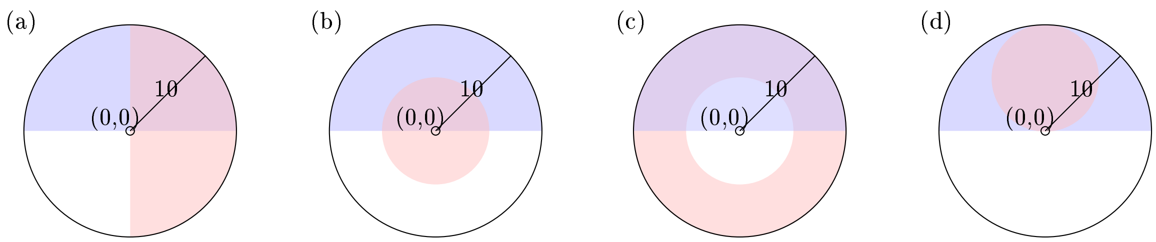
And for an event $F$, the conditional probability of $F$ given $E$ is:

$$P(F|E) = \int_F f(x|E)dx$$

Problem 1. Suppose that you toss a dart at a circular target of radius 10, and it lands at a point $(x, y)$ with $x, y$ each chosen uniformly at random within the target. Given that the dart lands in the upper half of the target, find the probability that:

(a) it lands in the right half of the target.
(b) its distance from the center is less than 5.
(c) its distance from the center is greater than 5.
(d) it lands within 5 of the point (0,5).

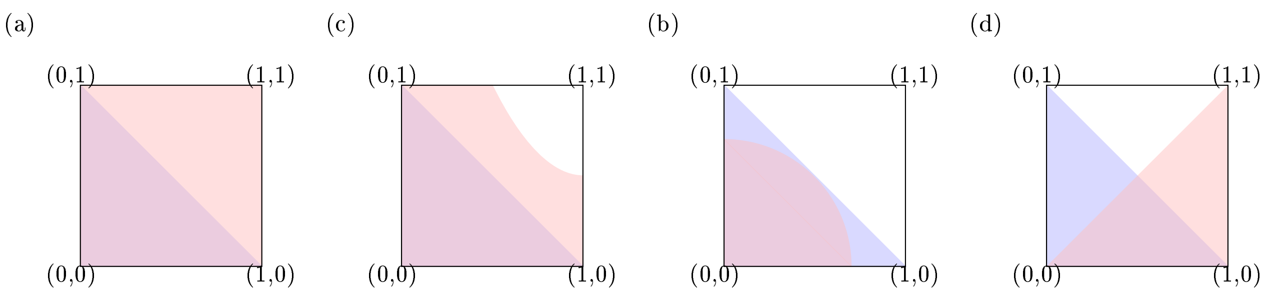
Let event $A$ be that the dart lands in the upper half of the target (shaded blue), and events described above be $A$, $B$, $C$ and $D$ (shaded red) respectively. Then $P(A|E)$ is the area of the intersection of $A$ and $E$ divided by the area of $E$.




$P(A|E) = 1/2$ $P(B|E) = 1/4$ $P(C|E) = 3/4$ $P(D|E) = 1/2$

Problem 2. Suppose you choose two numbers $x$ and $y$, each uniformly at random from the interval $[0, 1]$. Given that their sum lies in the interval $[0, 1]$, find the probability that:

(a) $|x - y| < 1$
(b) $xy < 1/2$
(c) $x^2 + y^2 < 1/4$
(d) $x > y$




$P(A|E) = 1$ $P(B|E) = 1$ $P(C|E) = \pi/4$ $P(D|E) = 1/2$

## 2 Independent Events

**Definition 1** The definition of independent events for continous sample spaces is analogous to the discrete case. Events $E$ and $F$ are independent if $P(E|F) = P(E)$ and $P(F|E) = P(F)$. Equivalently, $P(E \cap F) = P(E) \times P(F)$.

Problem 3. Suppose that in problem 1, if we don't have the condition that the dart lands in the upper half of the target, we still can assume it lands somewhere in the target. Which of the cases a, b, c, d in problem 1 are independent of the condition that the dart lands in the upper half of the target? Which of the cases a, b, c, d in problem 2 are independent of the condition that $x + y \in [0, 1]$?

| Problem 1: | | | Problem 2: | | |
|---|---|---|---|---|---|
| $P(A\|E) = 1/2$ | ✓ | $P(A) = 1/2$ | $P(A\|E) = 1$ | ✓ | $P(A) = 1$ |
| $P(B\|E) = 1/4$ | ✓ | $P(B) = 1/4$ | $P(B\|E) = 1$ | ✗ | $P(B) = 1/2 + ln(2)$ |
| $P(C\|E) = 3/4$ | ✓ | $P(C) = 3/4$ | $P(C\|E) = \pi/4$ | ✗ | $P(C) = \pi/8$ |
| $P(D\|E) = 1/2$ | ✗ | $P(D) = 1/4$ | $P(D\|E) = 1/2$ | ✓ | $P(D) = 1/2$ |

Problem 4. Let $x$ and $y$ each be chosen uniformly at random from the interval $[0, 1]$. Show that the events $x > 1/3$ and $y > 2/3$ are independent.

Let $A$ be the event that $x > 1/3$ and $B$ be the event that $y > 2/3$. Then:
$P(A) = 2/3$
$P(B) = 1/3$
$P(A \cap B) = 2/9 = P(A)P(B)$
And so the two events are independent.

## 3 Joint Density

**Definition 2** Let $X_1, X_2, \ldots, X_n$ be continuous random variables associated with an experiment, and let $\bar{X} = (X_1, X_2, \ldots, X_n)$. Then the *joint density function* is written $f(x_1, x_2, \ldots, x_n)$, and the joint cumulative distribution function is defined by:

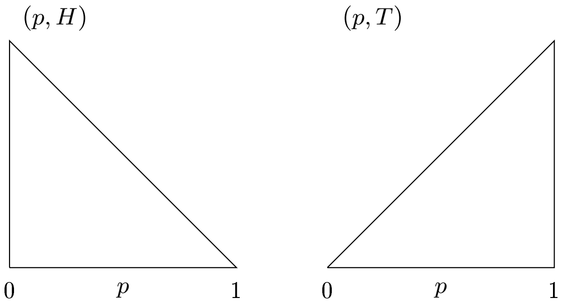
$$F(x_1, x_2, \ldots, x_n) = P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n).$$

We also have:

$$F(x_1, x_2, \ldots, x_n) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \ldots \int_{-\infty}^{x_n} f(t_1, t_2, \ldots, t_n) dt_1 dt_2 \ldots dt_n.$$

Problem 5*. You have a biased coin that comes up "tails" with probability $p$, where $p$ is drawn from $[1, 0]$ uniformly at random. The coin is tossed once. What is the probability density of $p$ given that the toss comes up "tails?".

The sample space is $\Omega = [0, 1] \times \{H, T\}$. We are interested in the crossection $\bar{X} = (p, T)$. The joint probability density can be desribed in these two cross-sections:




The overall probability of getting "tails" is 1/2. Suppose that you run the experiment of first picking $p$ and then tossing the coin many times, and record the probability $p$ during all the times that you get "tails." If you perform the experiment $10^6$ times, you expect to get $p$ between 0 and 0.01 about $10^4$ times, and $p$ between 0.5 and 0.51 about $10^4$ times. But of the former, only about 50 (for average $p$ between 0 and 0.1) will roll "tails", while of the latter, about 5050 will. So the amount of $p$ represented is proportional to $p$. Finally, we need to divide by the probability of getting "tails", 1/2. The distribution is:

$$f(p|T) = 2p.$$