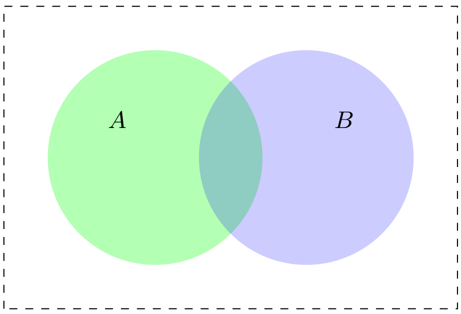
1 Conditional Probability and Independence

If A, B are events, what is the probability of A, given that B is true?



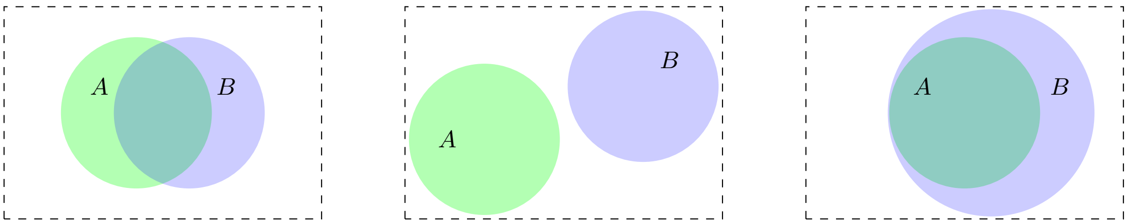
We are looking for the proportion of the probability of event B that is taken up by the intersection. We know we're in the blue circle. So what's the chance we're also in the green one?

$$\frac{P(A \cap B)}{P(B)}.$$

The *conditional probability of A given B* is written:

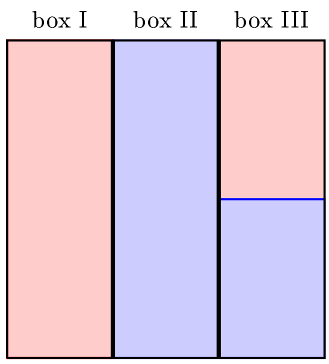
$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

The intersection of A and B can have many different sizes, the only limitation is that it can never be bigger than A, and it can never be bigger than B. And so without additional information it's hard to predict what $P(A|B)$ would be.



$P(A|B) = P(A \cap B)/P(B)$ $\qquad$ $P(A|B) = 0$ $\qquad$ $P(A|B) = P(A)$

Example 1 Recall the balls in boxes example. There are 2 red balls in box I, 2 blue balls in box II and 1 red and 1 blue ball in box III. Pick a box at random, and then pick a ball in that box at random. If the picked ball is blue, what's the chance the picked box is box II?



A: the picked ball is blue
B: the picked box is box II

$P(B) = 1/3,\ P(A) = 1/2$. But $P(A \cap B) = P(B) = 1/3$ since once we pick box II the ball must be blue. So:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3}.$$

A good way to illustrate this is on a square of area 1, as pictured on the right. We know that we picked a blue ball, so we're in the area shaded blue. Two thirds of this area is in the column corresponding to box II.

Definition 1 Two events A and B are *independent* if they both have positive probability and if:

$$P(A|B) = P(A),$$
$$P(B|A) = P(B).$$

Question: Can you show that if $P(A|B) = P(A)$ is true, then $P(B|A) = P(B)$ must be true as well?

Example 2 Draw a card from a full deck uniformly at random. What is the probability that it's red, given that it's an ace?

Example 3 A fair coin comes up heads. What is the probability that it comes up heads next time?

2 Useful Fact About Independent Events

$$\text{If A and B are independent, then } P(A \cap B) = P(A) \times P(B).$$

Theorem 1 *If $P(A) > 0$ and $P(B) > 0$, then A and B are independent if and only if $P(A \cap B) = P(A) \times P(B)$.*

Proof The "if and only if" statement in a theorem really says two things. Firstly, that A and B are independent only if $P(A \cap B) = P(A) \times P(B)$, i.e. if A and B are independent then $P(A \cap B) = P(A) \times P(B)$. And secondly, that A and B are independent if $P(A \cap B) = P(A) \times P(B)$. Let's prove the first statement.

Suppose that A and B are independent. Then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A).$$

Multiplying both sides by $P(B)$ results in:

$$P(A \cap B) = P(A) \times P(B).$$

The proof for $P(B|A)$ is analogous.

The second statement we need to prove is that if $P(A \cap B) = P(A) \times P(B)$, then A and B must be independent. We have:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \times P(B)}{P(B)} = P(A),$$

and so A and B must be independent. □

Question: Can you prove that if A and B are independent, then necessarily A and $\tilde{B}$ as well as $\tilde{A}$ and $\tilde{B}$ are independent?

3 Mutually Independent Events

A set of events $\{A_1, A_2, \ldots, A_n\}$ is *mutually independent* if for any subset of these events $\{A_i, A_j, \ldots A_m\}$, the probability of the intersection of this subset is the product of the individual probabilities:

$$P(A_i \cap A_j \cap \cdots \cap A_m) = P(A_i) \times P(A_j) \times \cdots \times A_m.$$

A set of events $\{A_1, A_2, \ldots, A_n\}$ is *pairwise independent* if any two events in the set are independent.

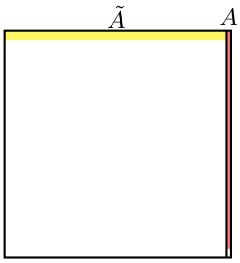
Any mutually independent set is pairwise independent by definition, but the opposite is not always true.

Example 4 Toss a coin twice. The following set of events is pairwise independent but not mutually independent. (Why?)

A: the first toss is heads.
B: the second toss is heads.
C: both tosses are the same.

4 Evaluating Conditional Probabilities

Example 5 Suppose that 0.5% of population carry a particular virus. The test for that virus is not perfect. Of those that do, 98% test positive. Of those that don't, 1% still test positive. If Alice tested positive, what is the probability that she carries the virus?



The yellow section corresponds to the probability that Alice does not carry the virus, but still tested positive. The red section correspond to the probability that Alice does carry the virus, and tested positive. The probability that she tests positive is the sum of the two. If:

A is the event that Alice carries the virus.
B is the probability that Alice tests positive.

We are interested in finding $P(A|B)$. We have that:

$P(A) = 0.005$
$P(B) = P(A \cap B) + P(\tilde{A} \cap B)$
$P(A \cap B) = 0.005 \times 0.98 = 0.0049$
$P(\tilde{A} \cap B) = 0.995 \times 0.01 = 0.00995$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.0049}{0.0049 + 0.00995} \simeq 0.3.$$

So even if Alice tested positive, there is about 0.7 probability that she does not carry the virus.

5 Joint Random Variables and Joint Distributions

If we roll two 6-sided dice, and X_1 is the outcome on the first one and X_2 is the outcome on the second, the combined outcome $\bar{X} = (X_1, X_2)$ is a*joint random variable.* It's sample space is the Cartesian product of the sample spaces of X_1 and X_2. Recall that we have already seen this in the first week, in the form of a table:

Table 1 The sample space $\Omega_{\tilde{X}} = \Omega_{X_1} \times \Omega_{X_2}$.

	⚀	⚁	⚂	⚃	⚄	⚅
⚀	(⚀,⚀)	(⚀,⚁)	(⚀,⚂)	(⚀,⚃)	(⚀,⚄)	(⚀,⚅)
⚁	(⚁,⚀)	(⚁,⚁)	(⚁,⚂)	(⚁,⚃)	(⚁,⚄)	(⚁,⚅)
⚂	(⚂,⚀)	(⚂,⚁)	(⚂,⚂)	(⚂,⚃)	(⚂,⚄)	(⚂,⚅)
⚃	(⚃,⚀)	(⚃,⚁)	(⚃,⚂)	(⚃,⚃)	(⚃,⚄)	(⚃,⚅)
⚄	(⚄,⚀)	(⚄,⚁)	(⚄,⚂)	(⚄,⚃)	(⚄,⚄)	(⚄,⚅)
⚅	(⚅,⚀)	(⚅,⚁)	(⚅,⚂)	(⚅,⚃)	(⚅,⚄)	(⚅,⚅)

In Example 5, let X_1 be 0 if Alice does not carry the virus, and 1 if she does. Let $X_2 \in (+, -)$ be the outcome of the test. The *joint distribution function* of $\tilde{X} = (X_1, X_2)$ is as follows.

$\Omega_{\tilde{X}} = \{(0, +), (1, +), (0, -), (1, -)\}$
$m((0, +)) = 0.995 \times 0.01 = 0.00995$
$m((1, +)) = 0.005 \times 0.98 = 0.0098$
$m((0, -)) = 0.995 \times 0.99 = 0.98505$
$m((1, -)) = 0.005 \times 0.02 = 0.0001$

It is not hard to check that these values add up to 1.