LECTURE 5 NOTES **Math 20 Fall 2014, Dartmouth College**

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1 Binomial coefficients

We saw that if we are picking j elements out of a set of n in order, there are

$$\underbrace{n \times (n-1) \times (n-2) \times \cdots \times (n-j+1)}_{j \text{ terms}} = \frac{n!}{j!}$$

ways to do so. Then if order doesn't matter a choice of abc is equivalent to, say, a choice of acb. In fact, any choice in order will be equivalent to $j!$ orders ($j! - 1$ other orders) of the same j elements.

Definition 1 The binomial coefficient $\binom{n}{j}$, sometimes called "n choose j" or "Newton's binomial" is the number of ways in which we can pick j elements out of a set of n. The value of the binomial coefficient is:

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}.$$

Can you evaluate these binomials?

$\binom{n}{0} =$

$\binom{n}{1} =$

$\binom{n}{2} =$

$\binom{n}{n-1} =$

$\binom{n}{n} =$

Notice that for any $0 \leq j \leq n$:

$$\binom{n}{j} = \binom{n}{n-j}.$$

There are as many ways to pick j element as there are ways to pick $n-j$ elements to NOT be in the chosen set.

Theorem 1 *For integers n and j, with $0 < j < n$, the binomial coefficients satisfy:*

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}.$$

Proof Let $A + \{1, 2, \ldots, n\}$. A choice of j elements of A can either include the number n or not. There are $\binom{n-1}{j}$ ways to NOT include it, since we then need to choose j elements from $\{1, 2, \ldots, n-1\}$. There are $\binom{n-1}{j-1}$ ways to include it, since we still need to choose $j-1$ elements from $\{1, 2, \ldots, n-1\}$. □

Question: What is $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$? It's the total number of subsets of an n-element set. There are 2^n subsets, since each element can either be included or not. So:

$$\sum_{j=0}^{n} \binom{n}{j} = 2^n.$$

2 Probabilities from combinations: poker

Suppose that you draw a hand of 5 cards out of a deck of 52. There are $\binom{52}{5}$ possible hands, and each one is equally likely. There are 13 ranks and 4 suits of cards.

What is the probability of getting four-of-a-kind? That means that we get four cards of the same rank and one other card. To calculate how many possible ways there are to get it, first choose the rank that we have four cards of. There are 13 options. The fifth card can be anything else, so any of the other 48 cards.

$$P(\text{four-of-a-kind}) = \frac{13 \times 48}{\binom{52}{5}}$$

A full house is a hand that has three cards of a matching rank and two cards of another, matching, rank. There are 13 possible ranks for the three, and $\binom{4}{3} = 4$ ways to choose the suits of the three cards. Then there are 12 ways to choose the other rank, and $\binom{4}{2} = 6$ ways to choose the suits. So:

$$P(\text{full house}) = \frac{13 \times 4 \times 12 \times 6}{\binom{52}{5}}$$

Three-of-a-kind is a hand that has three cards of a matching rank and two that don't match. Again, there are $13 \times \binom{4}{3}$ ways to pick the three matching cards. Then the other two have to come from the remaining 48, and not match. One way to do this is $\binom{48}{2} - 12 \times \binom{4}{2}$, hich takes any two cards and subtracts those of matching rank. Another way is to pick one of 48 cards, then one of 44 that have a different rank. However, that's picking the two cards in order, and they are equivalent. We need to divide by 2!.

$$P(\text{three-of-a-kind}) = \frac{13 \times 4 \times 48 \times 44/2}{\binom{52}{5}}$$

Two pair is a hand that has two pairs of matching rank an another fifth card. We pick a rank for the first pair, that's out of 13, the suits, that's $\binom{4}{3}$. We pick one of 12 ranks for the second pair, and suits again. The fifth card can be any of the remaining 44. However, the two pairs are interchangeable and we picked them in order. so we need to divide by 2!.

$$P() = \frac{13 \times 6 \times 12 \times 6 \times 44}{\binom{52}{5}}$$

If you evaluate these probabilities, you'll find out that two pair is much more likely than three-of-a-kind.

3 Bernoulli trials

Definition 2 A *Bernoulli trials process* is a sequence of n chance experiments such that:

1. Each experiment has two possible outcomeswhich we may call *success* and *failure*.
2. The probability p of success on each experiment is the same and is not affected by any knowledge of previous outcomes. The probability of failure is $1 - p$.

Theorem 2 *Given n Bernoulli trials with probability of success p, the probability of exactly j successes is:*

$$b(n, p, j) = \binom{n}{j} p^j (1-p)^{n-j}.$$

Proof Any particular sequence of successes anf failures of length n with exactly j successes occurs with probability $p^j(1-p)^{n-j}$, since every success contributes a p and every failure contributes a $1-p$. There are $\binom{n}{j}$ such sequences. □

Example 1 Roll a six-sided die four times. The probability that you roll a 6 exactly 1 time is:

$$b(4, 1/6, 1) = \binom{4}{1} \frac{1}{6} (\frac{5}{6})^3$$

Question: What's the probability that you roll a six AT LEAST once?

4 Binomial Distribution: first glance

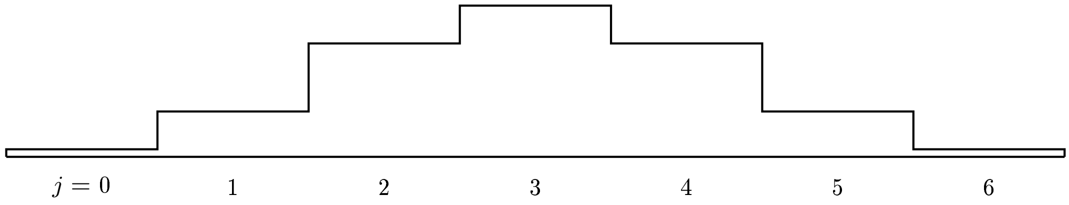
You might notice that binomial coefficients $\binom{n}{j}$ grow as j gets closer to $n/2$. For example, if you look at $n = 6$:

$\binom{6}{0} = 1$
$\binom{6}{1} = 6$
$\binom{6}{2} = 15$
$\binom{6}{3} = 20$
$\binom{6}{4} = 15$
$\binom{6}{5} = 6$
$\binom{6}{6} = 1$

So if we toss a fair coin 6 times, and let the random variable X count the number of times it shows heads, we get:

$$b(6, 1/2, j) = \binom{6}{j} \frac{1}{2^6}$$

which can be illustrated on a distribution diagram:



As n grows, the diagram will appear smoother, and more strongly centered around $n/2$. In general, even if p is not $1/2$, it will be centered around $p \times n$. The diagram below shows the distributions for $n = 40$, with $p = 1/2$ and $p = 0.2$.