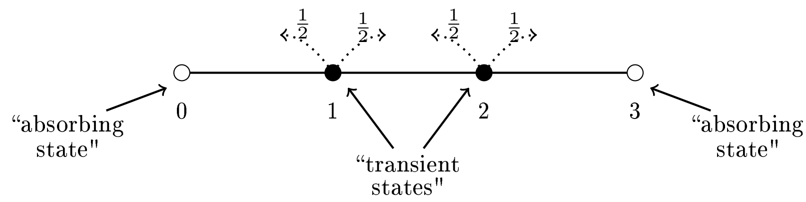
Lecture 22 **Math 20 Fall 2014, Dartmouth College**

Recall: Gambler's Ruin.



A *transient state* of a Markov chain is a state that the chain may visit a few times, but eventually will abandon it forever. An *absorbing state* is one that you never leave once you enter it.

Can you come up with a walk that has states that are neither transient nor absorbing?

The transition matrix for Gambler's Ruin on 4 nodes is:

$$P = \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For an absorbing MC on states $\{s_1, s_2, \ldots, s_M\}$ the transition matrix is in *canonical form* if the states are numbered so that the absorbing states come last. This matrix then looks like:

$$\left[\begin{array}{c|c} Q & R \\ \hline 0 & I \end{array}\right]$$

Where the lower right corner is an $r \times r$ identity matrix for all the states that you never leave once you enter, Q lists probabilities of going to another transient state from transient states, and R are probabilites of falling into absorbing states from transient states.

For all that, all we did was rearrange the states. So this is still a transition matrix. And if we raise it to power n, it will still list transition probabilities for n steps.

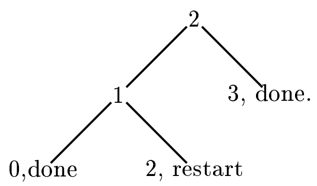
In particular, as times goes on we're less and less likely to still be in transient stats. so:

$$Q^n \to 0 \text{ as } n \to \infty.$$

In the example of Gambler's Ruin, the transition matrix in canonical form is:

$$\begin{matrix} 1 \\ 2 \\ 0 \\ 3 \end{matrix} \left[\begin{array}{cc|cc} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

How many times do you expect to visit node 1 if you start from node 2? This can be done on a tree diagram:



Start at 2, you go to either 3 or 1 each with probability 1/2. If you go to 3, it's game over. If you go to 1, you get to check off one visit to 1, and keep going. Then you go to either 0 or 2. If it's 0, it's game over. If it's 2, you're in the same situation you started from. So, if v_{21} is the expected number of visits to 1 if you start at 2:

$$v_{21} = \frac{1}{2} \times 1 + \frac{1}{4} \times v_{21}$$
$$\frac{3}{4} v_{21} = \frac{1}{2}$$
$$v_{12} = \frac{2}{3}.$$

Here's a wholesale version of the same calculation, using the magical efficiency of matrices. Let the *fundamental matrix* N be:

$$N = (I - Q)^{-1}$$

In the example, the fundamental matrix is:

$$N = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}^{-1} = \frac{1}{1 - \frac{1}{4}} \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}$$

The entries N_{ij} of the fundamental matrix are the expected number of times a walk that starts at ith transient state visits jth transient state. Bot for diagonal entries N_{ii} they count starting in the state as the first visit!

Having the expected number of times a site is visited is incredibly powerful - for example, we can find how long, on average the game will last by simply adding the number of times we expect to visit each node. So for each starting state, find the corresponding row of the fundamental matrix and add the entries in that row. This corresponds to multiplying the matrix by a vector with all entries 1.

If t_i is the expected duration of the game (*time to absorption*) if we start at ith transient state, then;

$$t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_{M-r} \end{pmatrix} = N \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Finally, if we start at ith transient state what is the probability of eventually falling into the jth absorbing state? We know how many times we expect to visit each transient state. Each of those visits carries some chance of falling into jth absorbing state, and these probabilities are listed in matrix R. Let the probability that if we start at ith transient state, we eventually fall into the jth absorbing state be B_{ij}.

$$B_{ij} = \sum_k N_{ik} R_{kj},$$

where k varies over transient states. The corresponding matrix is:

$$B = N \times R.$$

For the Gambler's ruin case,

$$B = \begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

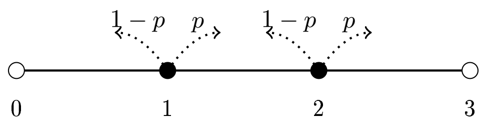
Exercises:

1. For the weather in the Land of Oz, change R into an absorbing state. The new transition matrix is then:

$$P = \begin{matrix} R \\ N \\ S \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

   Find the fundamental matrix N, and the vector t. What do the entries stand for?

2. Consider Gambler's Ruin on 4 nodes again, but this time the coin is biased and comes up in Bob's favor with probability p, and in Alice's favor with probability $1 - p$:



   Find the expected duration of the game and win probabilities for each starting state.