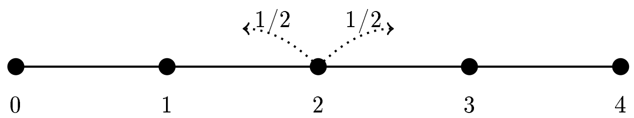
Lecture 20 **Math 20 Fall 2014, Dartmouth College**

A *simple random walk* on a graph is a random process $\{X(0), X(1), X(2), \dots\}$ in which the nodes of the graph form the state space, with some starting state $X(0)$ and for each $X(t)$, $X(t+1)$ can be any vertex adjacent to vertex $X(t)$ with equal probability. In other words, imagine a token that starts out on some node in a graph and at each time interval moves to a random neighbor of that node. On a path as pictured, if you're on one of the nodes at ends you only have one way to go. If you're on a middle node, move either right or left with equal probability.

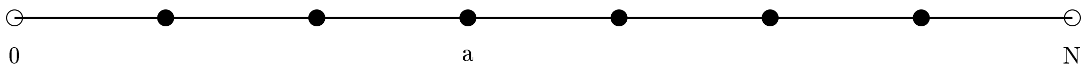


The walk is not "simple" if the probabilities of moving to different neighbors are not equal.

1 Gambler's Ruin

Alice has \$$a$ and Bob has \$$b$, with $a + b = N$. They will flip a coin. If it comes up heads, Alice hives Bob \$1. If it comes up tails, Bob gives Alice \$1. They will repeat this until someone has \$$N$. What is the probability that Alice wins?

The amount of money Alice has can be represented as a simple random walk on a path with nodes from 0 to N, starting at a. What is the chance that the walk reaches N before it reaches 0?



A good way to evaluate this, is to keep N fixed and find the probability for various as. Let $P(x)$ be the probability that the walk that stats at node x reaches N before it reaches 0.

$P(0) = 0$
$P(1) = 1$
For $0 < x < N$, $P(x) = P(x-1)/2 + P(x+1)/2$.

This last relation comes from the fact that this probability must be the same as the expected probability after the first move. And after first move we're at $x - 1$ or $x + 1$ each with probability $1/2$. So, if $0 < x < N$:

$2P(x) = P(x-1) + P(x+1)$
$P(x) - P(x-1) = P(x+1) - P(x) = \dots = P(1) - P(0)$

So the difference between each probability and the next is the same, the function must be linear! Since the difference is $P(1)$, we get:

$P(x) = xP(1)$
$P(N) = 1 = NP(1)$
$P(1) = 1/N$
$P(x) = x/N$.

So a simple random walk on a path with nodes numbered from 0 to N, starting at x hits N before it hits 0 with probability x/N. What if Alice and Bob have the same amount of money to start with? What property of the graph does that correspond to?

2 Time to Absorption

In the Gambler's Ruin, 0 and N are called *absorbing states*, since once you hit one of them, you stay there and the game ends. What is the *time to absorption,* i.e. the expected amount of moves that the game will last?

Let $T(x)$ be the expected time to absorption of a simple random walk on a path with nodes numbered from 0 to N, starting at a node x. Then:

$T(0) = 0$
$T(N) = 0$
For $0 < x < N$, $T(x) = 1 + T(x-1)/2 + T(x+1)/2$.

The function T is unique, we prove it below.

Claim: There is only one function f on $\{0, 1, 2, \dots, N\}$ such that $f(0) = f(N) = 0$ and for any $0 < x < N$,

$$f(x) = 1 + \frac{1}{2}f(x-1) + \frac{1}{2}f(x+1).$$

Proof: Suppose that there are TWO such functions, f and g. We will show that $f(x) - g(x) = 0$, so must in fact be the same function. Consider the function $h(x) = f(x) - g(x)$. We have:

$h(0) = 0$
$h(N) = 0$
For $0 < x < N$, $h(x) = \frac{1}{2}h(x-1) + \frac{1}{2}h(x+1)$.

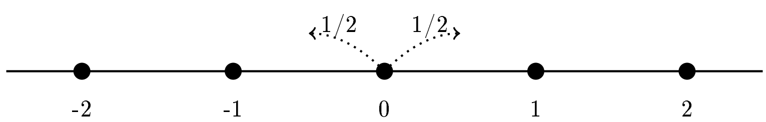
Now we need to show that this implies $h(x) = 0$ for all nodes x. Suppose that $0 < a$ is the smallest node such that $h(a) \neq 0$. Since $h(a) = \frac{1}{2}h(a-1) + \frac{1}{2}h(a+1) = 0 + \frac{1}{2}h(a+1)$, $h(a+1)$ has to be even further from 0. Similarly, $h(a+2)$ has to be even further from 0 than that, and so on, until we reach $h(N) \neq 0$, which contradicts the assumptions. We conclude therefore that such situation is impossible and $f(x) = g(x)$ for all nodes x. □

Since the function T is unique, if we guess one we know it's the only on that fits. Looking at $N = 2,\ N = 3$ we get an impression that function could be $x(N-x)$. Let's try it.

$RHS = 1 + (x-1)(N-x+1)/2 + (x+1)(N-x-1)/2$
$= 1 + (x-1)(N-x)/2 + (x+1)(N-x)/2 + (x-1)/2 - (x+1/2)$
$= 1 + (x-1-x+1)(N-x)/2 + (x-1-x-1)/2 = x(N-x) = LHS.$

The expected time to absorption of this walk is $x(N-x)$.

3 Walk on the Integers



Take a simple random walk on the integers, the line stretches out to infinity on both sides. If you start at 0, what is the probability distribution of your position after k steps?

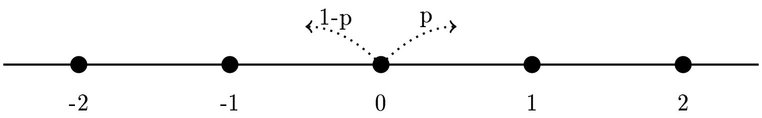
If k is even:

$$m(i) = \begin{cases} \frac{1}{2^k}\binom{k}{(k+i)/2} \text{ if } i \text{ is even and } -k \leq i \leq k \\ 0 \text{ if } i \text{ otherwise.} \end{cases}$$

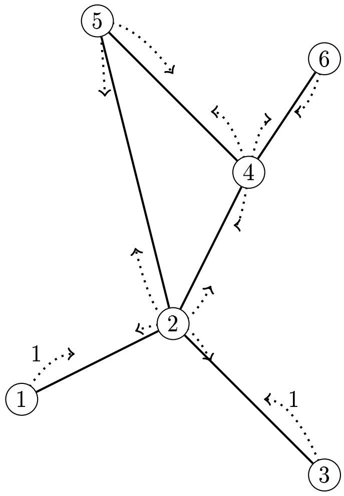
If k is odd:

$$m(i) = \begin{cases} \frac{1}{2^k}\binom{k}{(k+i)/2} \text{ if } i \text{ is odd and } -k \leq i \leq k \\ 0 \text{ if } i \text{ otherwise.} \end{cases}$$

What if the walk is not simple, but instead wherever you are the probability of a step to the right is p and a step to the left is q?



4 Arbitrary Networks



Random Walks are an example of a *Markov chain.* We will look at Markov Chains more closely next week, but here is the gist of it: Markov chains are random processes with a set state space, where the distribution of where it goes next only depends on the current position, not on the entire history. *Markov chains* are often presented in the form of *transition matrix.* If M the transition matrix for a Markov chain, entry T_{ij}, i.e. the one in row i and comlumn j is the probability that if we are in state i, we will be in state j on the next step. For the simple random walk on the graph pictured above, the transition matrix is:

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$