Lecture 19 **Math 20 Fall 2014, Dartmouth College**

1 A Look Back at Joint Random Variables

Suppose that X, Y are two discrete r.v.s and we would like to know how they interact. Then the joint probability distributions is:

$$m_{X,Y}(x,y) = P(X = x, Y = y).$$

Just like with single variables,

$$\sum_{x \in \Omega_X} \sum_{y \in \Omega_Y} m_{X,Y}(x,y) = 1.$$

To collapse it back to the distribution of X alone, take:

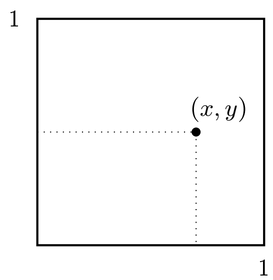
$$P(X = x) = \sum_{y \in \Omega_Y} m_{X,Y}(x,y).$$

In the continuous case, it's often helpful to look at the joint cumulative probability distributions first.

$$F_{X,Y}(x,y) = P(X \le x, Y \le y),$$

then just the density of X is $f_X(x) = \frac{d}{dx} F_X(x)$ and the density of Y is $f_Y(y) = \frac{d}{dy} F_Y(y)$, we have:

$$f_{X,Y}(x,y) = \frac{d}{dx}\frac{d}{dy} F_{X,Y}(x,y).$$



Example: If both X and Y are sampled uniformly at random from $[0,1]$, then $F_{X,Y}(x,y) = xy$. Then:

$$f_{XY} = \frac{d}{dx}\frac{d}{dy} xy = \frac{d}{dx} x = 1.$$

For any $f_{X,Y}(x,y)$:

$$\int_{\Omega_X} \int_{\Omega_Y} f_{X,Y}(x,y)dydx = 1.$$

If X and Y are independent, then:

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)$$

and

$$f_{X,Y}(x,y) = \frac{d}{dx}\frac{d}{dy}(F_X(x)F_Y(y)) = \frac{d}{dx}F_X(x)\frac{d}{dy}F_Y(y) = f_X(x)f_Y(y).$$

Example 1 Let X, Y be independent exponential random variables with:

$$f_X(x) = \begin{cases} \lambda_1 e^{-\lambda_1 x} \text{ if } x \in [0,\infty) \\ 0 \text{ otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \lambda_2 e^{-\lambda_2 y} \text{ if } y \in [0,\infty) \\ 0 \text{ otherwise} \end{cases}$$

What is the probability that $X \le Y$?

Since the variables are independent, we have:

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2 y} \text{ if } X, Y \ge 0,$$

and 0 otherwise. So we need to evaluate:

$$P(X \ge Y) = \int_0^\infty \int_0^y f_{X,Y}(x,y)dxdy = \int_0^\infty \int_0^y \lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2 y} dxdy$$

The first step is to notice that part of the function does not depend on x:

$$\int_0^\infty \left(\int_0^y \lambda_1 e^{-\lambda_1 x} dx\right) \lambda_2 e^{-\lambda_2 y} dy = \int_0^\infty \left(1 - e^{-\lambda_1 y}\right) \lambda_2 e^{-\lambda_2 y} dy = \int_0^\infty \lambda_2 e^{-\lambda_2 y} dy - \int_0^\infty e^{-\lambda_1 y} \lambda_2 e^{-\lambda_2 y} dy$$

The first of the two integrals is just an integral over an entire sample space of an exponential density, so it's 1. The other one we can transform into an integral over exponential density with parameter $\lambda_1 + \lambda_2$:

$$P(X \leq Y) = 1 - \lambda_2 \int_0^\infty e^{-(\lambda_1+\lambda_2)y} dy = 1 - \frac{\lambda_2}{\lambda_1+\lambda_2} \int_0^\infty (\lambda_1+\lambda_2) e^{-(\lambda_1+\lambda_2)y} dy = 1 - \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

Example 2 (Random Variables That Are Not Independent)

Suppose that X is picked uniformly at random from $(0,1)$ and Y is picked uniformly at random from $(0, 1/x)$.

- What is the sample space $\Omega_{X,Y} = \Omega_X \times \Omega_Y$?
- What is the joint density $f_{X,Y}(x,y)$?
- Check that $\int_{\Omega_X} \int_{\Omega_Y} f_{X,Y}(x,y) dy dx = 1$.
- What is $F_{X,Y}(a,b)$?

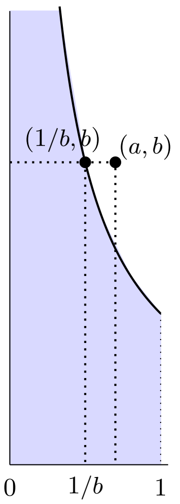
$\Omega_X = (0,1)$. Any positive number can appear as y, so $\Omega_Y = (0, \infty)$. The joint density function can be found as follows: once we have a value x for X, the density $f_{Y|x}(y)$ is x, since y is picked uniformly from an interval of length $1/x$. Then, since the conditional density is $f_{Y|x}(y) = f_{X,Y}(x,y)/f_X(x)$, we have:

$$f_{X,Y}(x,y) = f_{Y|x}(y) f_X(x) = x \times 1 = x \text{ if } y \leq 1/x,\ 0 \text{ otherwise.}$$

Let's now check the integral over the entire sample space:

$$\int_{\Omega_X} \int_{\Omega_Y} f_{X,Y}(x,y) dy dx = \int_0^1 \int_0^{1/x} x dy dx = \int_0^1 (x/x) dx = \int_0^1 1 dx = 1.$$

Finding $F_{X,Y}(a,b)$ is a little more involved. The area where the density is not 0 is marked in blue.



On the area marked in blue, $f_{X,Y}(x,y) = x$. We need to evaluate the integral of $f_{X,Y}(x,y)$ over the rectangle $(X < a, Y < b)$. For $x < 1/b$, the integral goes from 0 to b. Between $1/b$ and a, it goes from 0 to $1/x$. So:

$$F_{X,Y}(a,b) = \int_0^{1/b} \int_0^b x dy dx + \int_{1/b}^a \int_0^{1/x} x dy dx$$

$$= \int_0^{1/b} (xb) dx + \int_{1/b}^a 1 dx = 1/2b + (a - 1/b) = a - 1/2b$$

The case $a < 1/b$ is simpler, we only need to integrate:

$$\int_0^a \int_0^{1/b} x dy dx = \int_0^a x/b dx = a^2/2b.$$