1 Expected values of discrete random variables

Definition 1 Let X be a numerically-valued discrete r.v. with sample space Ω and distribution function $m(x)$. If $\sum_{x\in\Omega} x \times m(x)$ converges absolutely, then::

$$E(X) = \sum_{x\in\Omega} x \times m(x) \text{ is the "expected value of } X\text{."}$$

If it doesn't converge absolutely, X has no expected value.

Recall: $\sum_{x\in\Omega} x \times m(x)$ converges absolutely if and only if $\sum_{x\in\Omega} |x \times m(x)| < \infty$.

Example 1 Roll a six-sided die. If the result is even, you win the amount in $ equal to the result. If it's odd, you lose the amount in $ equal to the result. What are your expected winnings?

$$\sum_{x\in\Omega} x \times m(x) = (-1) \times 1/6 + 2 \times 1/6 + (-3) \times 1/6 + 4 \times 1/6 + (-5) \times 1/6 + 6 \times 1/6 = 1/2.$$

You would expect to win 50 cents.

Example 2 This example is called the *St. Petersburg Paradox.* Flip a coin until the first time it comes up "tails." If comes up "tails" for the first time on the nth flp, you win $\$2^n$.

$$\sum_{x\in\Omega} x \times m(x) = 2 \times \frac{1}{2} + 4 \times \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} 2^n \times \frac{1}{2^n} = \sum_{n=1}^{\infty} 1 = \infty$$

The winnings don't have an expected value. How much would you be willing to pay to play that game?

From now on, we will assume that the random variables we deal with all have expected values.

2 Expected values of functions and sums of discrete random variables

If $Y = g(X)$ is a function of random variable X, then it's also a random variable. Therefore:

$$E(Y) = \sum_{x\in\Omega} g(x) \times m(x)$$

In particular, if $Y = cX$ for some constant c, then $E(Y) = \sum_{x\in\Omega} c \times x \times m(x) = c\sum_{x\in\Omega} x \times m(x)$. So:

$$E(cX) = cE(X).$$

Theorem 1 *For any discrete random variables X, Y with expected values $E(X)$, $E(Y)$,*

$$E(X + Y) = E(X) + E(Y).$$

Proving the last equation may be done as follows:

$$E(X+Y)=\sum_j\sum_k(x_j+y_k)P(X=x_j,Y=y_k)=\sum_j\sum_k x_jP(X=x_j,Y=y_k)+\sum_j\sum_k y_kP(X=x_j,Y=y_k)$$

By definition of conditional probability:

$$P(X=x_j,Y=y_k)=P(X=x_j)P(Y=y_k|X=x_j)$$

$$P(X=x_j,Y=y_k)=P(Y=y_k)P(X=x_j|Y=y_k)$$

And so:

$$E(X+Y)=\sum_j x_jP(X=x_j)\underbrace{\sum_k P(Y=y_k|X=x_j)}_{1}+\sum_k y_kP(Y=y_k)\underbrace{\sum_j P(Y=y_k|X=x_j)}_{1}$$

$$=\sum_j x_jP(X=x_j)+\sum_k y_kP(Y=y_k)=E(X)+E(Y).$$

Example 3 Roll two six-sided dice. Let X and Y be the two outcomes.

$$E(X+Y)=2\times\frac{1}{36}+3\times\frac{2}{36}+4\times\frac{3}{36}+5\times\frac{4}{36}+6\times\frac{5}{36}+7\times\frac{6}{36}+8\times\frac{5}{36}+9\times\frac{4}{36}+10\times\frac{3}{36}+11\times\frac{2}{36}+12\times\frac{1}{36}=7$$

This can also be done in a more straightforward way:

$$E(X)=E(Y)=1\times\frac{1}{6}+2\times\frac{1}{6}+3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6}=3.5,$$

and so:

$$E(X+Y)=E(X)+E(Y)=3.5+3.5=7.$$

3 Bernoulli trials

We do a sequence of n Bernoulli trials, each with probability of success p, and the random variable X is the number of successes. What is the expected value of X?

Each trial is a random variable in its own right, that takes value 1 with probability p and value 0 with probability $1-p$. Then the random variable X is the sume of the outcomes of the separate trials:

$$X=X_1+X_2+\cdots+X_n$$

$$E(X)=E(X_1)+E(X_2)+\cdots+E(X_n)$$

And for each X_i, $E(X_i)=0\times(1-p)+1\times p$. So:

$$E(X)=n\times p.$$

4 Poisson distribution

We'd expect the Poisson distribution with parameter λ to also have a mean at $n\times p=\lambda$. Here's a derivation:

$$E(X)=\sum_{k=0}^{\infty}k\times e^{-\lambda}\frac{\lambda^k}{k!}=0+\sum_{k=1}^{\infty}k\times e^{-\lambda}\frac{\lambda^k}{k!}=\sum_{k=1}^{\infty}e^{-\lambda}\frac{\lambda^k}{(k-1)!}=\lambda\sum_{k=1}^{\infty}e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!}=\lambda\underbrace{\sum_{k=0}^{\infty}e^{-\lambda}\frac{\lambda^k}{k!}}_{1}=\lambda$$

5 Expectation of the product of independent r.v.s

If X, Y are independent, then:

$$E(X \times Y) = E(X) \times E(Y).$$

Proof

$$E(X \times Y) = \sum_j \sum_k x_j \times y_k P(X = x_j, Y = y_k) = \sum_j \sum_k x_j \times y_k P(X = x_j) \times P(Y = y_k),$$

by independence. Further:

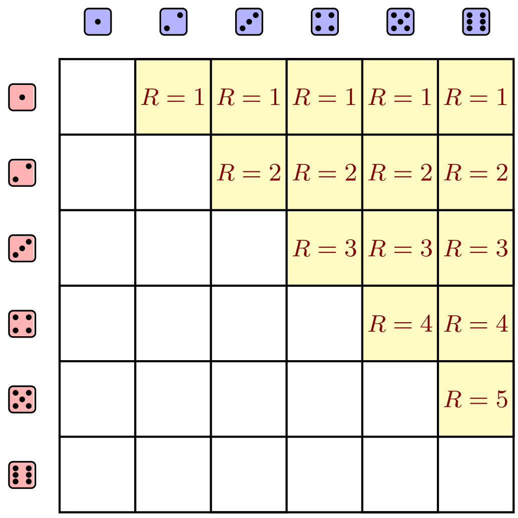
$$\sum_j \sum_k x_j \times y_k P(X = x_j) \times P(Y = y_k) = \sum_j x_j P(X = x_j) \times \sum_k y_k P(Y = y_k) = E(X) \times E(Y)$$

6 Conditional expectation

If F is an event, the *conditional expectation of X given F* is:

$$E(X|F) = \sum_{x \in \Omega} x \times P(X = x|F).$$

Example 4 Roll two six-sided dice, one red and one blue. Let random variables R and B be the outcome on the red and the blue die respectively. If the event F is the condition $R < B$, What is $E(R|F)$?



There are 15 combinations (marked yellow) that satisfy the condition $R < B$. The conditional expected value is as follows:

$$E(R|F) = 1 \times \frac{5}{15} + 2 \times \frac{4}{15} + 3 \times \frac{3}{15} + 4 \times \frac{2}{15} + 5 \times \frac{1}{15} = \frac{35}{15}$$